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GAS-DYNAMIC APPROACH IN THE NONLINEAR THEORY OF ION ACOUSTIC WAVES IN A PLASMA: AN EXACT SOLUTION

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An exact solution of the problem of the acoustic wave structure in a plasma is obtained. Both plasma component are treated as gases with specified initial temperatures and adiabatic exponents. The system of equations describing the wave profile is solved using an original method consisting of reducing the system to the Bernoulli equation. A numerical example of the obtained general solution of the problem of the wave profile for arbitrary parameters is given. Curves are constructed that bound the region of existence of a stationary solitary ion acoustic wave in the parameter space.

Key words: *plasma, ion acoustic wave, nonlinear theory.*

Introduction. The propagation of ion acoustic waves, which is one of the basic wave processes in a plasma, has been studied for several decades. Nonlinear theory for these waves was first considered in [1, 2], where their basic features were studied using the method of mechanical analogy (in foreign literature, the pseudo-potential method). It has been established that stationary waves can exist in the form of a periodic or solitary wave and that the wave velocity is bounded from above by a value approximately 1.58 times exceeding the linear ion-acoustic velocity. An exact expression for the limiting velocity was found in [3]. It has been assumed [1–3] that the ion plasma component is cold and that the electron plasma component is isothermal and inertialess.

Subsequently, the nonlinear theory has been developed in numerous studies taking into account the influence of ion temperature, the presence of two or more sorts of ions, including negative ions, the presence of two groups of electrons at different temperatures, the inertia of electrons, etc. (for more details see [4]).

In the papers cited above and in most other papers, it was assumed that the hot plasma components are involved by the wave in an isothermal process, i.e., that their temperature is constant. This simplification ignores the question of the external source or sink of thermal energy since an isothermal process is necessarily accompanied by the energy input due to plasma flow compression and the energy release due to its unloading.

Thus, for the description of nonlinear waves in plasma, models considering the process adiabatic are more realistic. This approach allows one to take into account temperature variations in different wave phases and the effect of this variation on the formation and properties of the wave.

Recently, a gas-dynamic approach has been used to study ion acoustic and dust acoustic waves [5–8]. The nonlinear equations describing the structure of the waves were analyzed in [5–8] using an adiabatic approach, in which the ion or dust plasma component was a gas. The equation of state for the gas was taken in the form of an adiabat with an arbitrary parameter γ_+ in the range $\gamma_+ \in [1; 3]$. The boundaries of the regimes and the limiting wave velocities were determined. However, the general exact solution of the problem of the wave profile was not obtained in [5–8] (as is known, the properties of solutions can be analyzed without solving the equations) although an exact solution for the particular case of cold ions with an adiabatic exponent of the electron component $\gamma_- = 2$ was given in [5].

The present paper gives, for the first time, an exact solution of the problem of the ion acoustic wave structure in a plasma using the gas-dynamic approach for arbitrary values of $\gamma_\pm$ in the range $\gamma_\pm \in [1; 3]$.

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1. Initial Equations and Notation. We consider an unbounded, noncollosional, and homogeneous plasma which contains only electrons and single-charged ions. The parameters of the electron component of the unperturbed plasma will be denoted as follows: m_- is the mass of the particles (the electrons are assumed to be inertialess, i.e., $m_- \to 0$), $e < 0$ is the particle charge, T_{0-} is the temperature, γ_- is the adiabatic exponent, and P_- is the pressure; the parameters of the ion component: m_+ is the mass of the particles, $-e > 0$ is the particle charge, T_{0+} is the temperature, γ_+ is the adiabatic exponent, P_+ is the pressure, and v_+ is the velocity. Because the concentration of particles of both charge signs in the unperturbed plasma are quasineutral, it follows that $n_{0\pm} = n_0$. The parameters perturbed in the wave will be written without the subscript 0.

The dynamics of the ion plasma component is described by the following one-dimensional equations:

— the continuity equation

$$\frac{\partial n_+}{\partial t} + \frac{\partial\,(n_+ v_+)}{\partial x} = 0; \tag{1}$$

— the equation of motion

$$m_+\Big(\frac{\partial v_+}{\partial t} + v_+\,\frac{\partial v_+}{\partial x}\Big) = e\,\frac{\partial \varphi}{\partial x} - \frac{1}{n_+}\,\frac{\partial P_+}{\partial x}; \tag{2}$$

— the Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n_+ - n_-) \tag{3}$$

(φ is the electrostatic potential).

System (1)–(3) is supplemented by the equation of state of the ion gas for the adiabatic process:

$$P_+ = kT_{0+} n_0 (n_+/n_0)^{\gamma_+} \tag{4}$$

(k is Boltzmann's constant). In the Poisson equation (3), it is necessary to take into account the contribution of the electron component, which, similarly to (4), is described assuming that the process is adiabatic. Then, writing the equation of the dynamics of the electron gas in the form of (2), where the electron mass tends to zero because of no inertia, we can derive an equation that relates the electron concentration and the electrostatic potential. As a result, we obtain (see [9])

$$n_- = n_0\Big(1 - \frac{\gamma_- - 1}{\gamma_-}\,\frac{e\varphi}{kT_{0-}}\Big)^{1/(\gamma_- - 1)}. \tag{5}$$

It is easy to show that as $\gamma_- \to 1$, Eq. (5) becomes an exponential Boltzmann distribution.

We introduce the following notation: λ_D is the Debye length, ω_+ is the ion plasma frequency, v_s is the linear ion-acoustic velocity:

$$\lambda_D = \sqrt{\frac{\gamma_- kT_{0-}}{4\pi n_0 e^2}}, \qquad \omega_+ = \sqrt{\frac{4\pi n_0 e^2}{m_+}}, \qquad v_s = \sqrt{\frac{\gamma_- kT_{0-}}{m_+}}.$$

For Eqs. (1)–(5), it is convenient to introduce the following normalizations:

$$n_\pm = n_0 n'_\pm, \quad T_\pm = T_{0-}T'_\pm, \quad \varphi = (\gamma_- kT_{0-}/e)\varphi', \quad v_\pm = v_s v'_\pm, \quad x = \lambda_D x', \quad t = \omega_+^{-1} t'.$$

It should be noted that the dimensional potential φ and the dimensionless potential φ' have opposite signs since $e < 0$. Below, the primes at the dimensionless quantities are omitted.

Thus, using the equation of state, system (1)–(3), (4), (5) can be written in dimensionless form

$$\frac{\partial n_+}{\partial t} + \frac{\partial\,(n_+ v_+)}{\partial x} = 0,$$
$$\frac{\partial v_+}{\partial t} + v_+\,\frac{\partial v_+}{\partial x} = \frac{\partial \varphi}{\partial x} - \frac{\tau}{\gamma_+ - 1}\,\frac{\partial}{\partial x}\,(n_+^{\gamma_+ - 1}), \tag{6}$$
$$\frac{\partial^2 \varphi}{\partial x^2} = n_+ - [1 - (\gamma_- - 1)\varphi]^{1/(\gamma_- - 1)},$$

where $\tau = \gamma_+ T_{0+}/(\gamma_- T_{0-})$.

2. Stationary Solution of the Equations. We consider a stationary ion acoustic wave propagating at dimensionless velocity M (M is the Mach number) in the x direction. For this, we introduce the self-similar variable

$$\xi = x - \mathrm{M}t, \qquad \frac{\partial}{\partial t} = -\mathrm{M}\,\frac{d}{d\xi}, \qquad \frac{\partial}{\partial x} = \frac{d}{d\xi}.$$

This means transition from the laboratory frame of reference to a new system moving together with the wave. As a result, the system of partial differential equations (6) reduces to the following system of ordinary differential equations:

$$-\mathrm{M}\frac{dn_+}{d\xi} + \frac{d\,(n_+ v_+)}{d\xi} = 0,$$

$$-\mathrm{M}\frac{dv_+}{d\xi} + v_+\,\frac{dv_+}{d\xi} = \frac{d\varphi}{d\xi} - \frac{\tau}{\gamma_+ - 1}\,\frac{d}{d\xi}\,(n_+^{\gamma_+ - 1}); \tag{7}$$

$$\frac{d^2\varphi}{d\xi^2} = n_+ - [1 - (\gamma_- - 1)\varphi]^{1/(\gamma_- - 1)}. \tag{8}$$

Before solving this system, we consider the well-known methods of the solution of similar systems. The most widely used method is the pseudo-potential method [1, 2], in which the equations of continuity and motion are integrated, the dependences of the ion concentration and velocity on the potential are obtained from the integrals, and the dependences obtained are substituted into the Poisson equation. The result is an autonomous second-order differential equation for the potential, which has the form of the equation of motion for an oscillator in a one-dimensional pseudo-potential with the electrostatic potential used as a pseudo-coordinate and the coordinate as pseudo-time. Another method was developed in [10] and consists of eliminating the ion concentration and potential from the system of equations and reducing the system to a second-order equation for the ion velocity. This equation is also solved and analyzed using the pseudo-potential method, but the pseudo-coordinate is the ion velocity. There is also a method described in [11, 12], according to which the potential and ion concentration are eliminated from the equations, and the system is thus reduced to a third-order equation for the velocity. The obtained equation is studied using a phase portrait. We were unable to apply the above-listed methods to the solution of Eqs. (7) and (8).

However, the given equations can be solved using the method developed for the solution of the problem of the structure of longitudinal waves of a spatial charge in a neutralized electron beam [13]. In this method, the potential and ions velocity are eliminated from the system of equations and the latter is reduced to an autonomous second-order equation for the ion concentration, which, in turn, is easily reduced to the well-known differential Bernoulli equation.

Thus, integration of the continuity equation subject to the condition $\lim\limits_{v_+ \to 0} n_+ = 1$ yields

$$n_+ = \mathrm{M}/(\mathrm{M} - v_+). \tag{9}$$

It should be noted that in the frame of reference attached to the wave, the unperturbed plasma moves at a velocity $-\mathrm{M}$ in the opposite direction. Therefore, in this case, it is more reasonable at first glance to use the condition $\lim\limits_{v_+ \to -\mathrm{M}} n_+ = 1$ (see, for example, [14, Sec. 8.2.3]). However, for the unperturbed plasma which is motionless in the laboratory frame of reference, it is more correct to use the condition $\lim\limits_{v_+ \to 0} n_+ = 1$ (see [1–3]) since in conversion to the new system of reference, the boundary on which this condition is specified also moves at a velocity $-\mathrm{M}$. In other words, the unperturbed plasma does not intersect this boundary.

We solve Eq. (9) for v_+:

$$v_+ = \mathrm{M}(1 - 1/n_+). \tag{10}$$

Integrating the equation of motion subject to the conditions $\lim\limits_{v_+ \to 0} n_+ = 1$ and $\lim\limits_{v_+ \to 0} \varphi = 0$, we have

$$-\mathrm{M}v_+ + \frac{v_+^2}{2} = \varphi + \frac{\tau}{\gamma_+ - 1}\,(1 - n_+^{\gamma_+ - 1}). \tag{11}$$

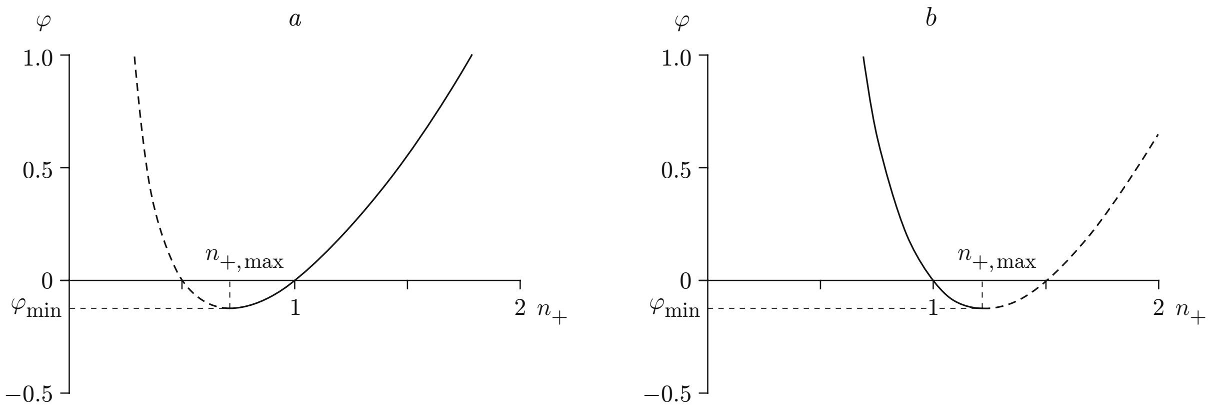

Fig. 1. Curve of $\varphi(n_+)$ (12) for $\tau = 1$, $\gamma_+ = 3$, and $\mathrm{M} = 0.5$ (a) and 1.5 (b); the dashed curves are the branches of the curves that have no physical meaning.

Substituting (10) into (11) and solving the resulting equation for the potential φ, we obtain

$$\varphi = \frac{\tau}{\gamma_+ - 1}\,(n_+^{\gamma_+ - 1} - 1) + \frac{\mathrm{M}^2}{2}\Big(\frac{1}{n_+^2} - 1\Big). \tag{12}$$

We note that an analytical solution of (12) for the concentration n_+ for an any adiabatic exponent γ_+ seems impossible, which explains the impossibility of using the methods considered above.

The plot of the relation $\varphi(n_+)$ (12) is a curve with a minimum. In addition, Eq. (12) should necessarily have the root $n_+ = 1$. This condition, together with expression (5), provides for the satisfaction of the quasineutrality condition for the unperturbed plasma. Typical curves of $\varphi(n_+)$ are presented in Fig. 1. In the case presented in Fig. 1a, the right branch passes through the quasineutrality point, and in the case presented in Fig. 1b, the left branch passes through it. The branches intersecting the abscissa at different points are nonphysical (the dashed curves in Fig. 1).

To determine the position of the minimum of the dependence $\varphi(n_+)$, we find its derivative and equate it to zero:

$$\frac{d\varphi}{dn_+} = -\frac{\mathrm{M}^2 - \tau n_+^{\gamma_+ + 1}}{n_+^3} = 0. \tag{13}$$

As a result, we obtain

$$n_{+,\max} = (\mathrm{M}^2/\tau)^{1/(\gamma_+ + 1)}; \tag{14}$$

$$\varphi_{\min} = \frac{\tau}{\gamma_+ - 1}\Big[\Big(\frac{\mathrm{M}^2}{\tau}\Big)^{(\gamma_+ - 1)/(\gamma_+ + 1)} - 1\Big] + \frac{\mathrm{M}^2}{2}\Big[\Big(\frac{\mathrm{M}^2}{\tau}\Big)^{-2/(\gamma_+ + 1)} - 1\Big].$$

It should be noted that in (14) the ion concentration is denoted by the subscript max since the ion density is maximal for the minimum dimensionless electrostatic potential (see below).

From (14) it follows that for $\mathrm{M} < \sqrt{\tau}$ only the right branch of the curve of $\varphi(n_+)$ has a physical meaning (see Fig. 1a), and for $\mathrm{M} > \sqrt{\tau}$ only the left branch has a physical meaning (see Fig. 1b).

In the subsequent analysis, we will also need the second derivative of (12):

$$\frac{d^2\varphi}{dn_+^2} = \frac{3\mathrm{M}^2 + (\gamma_+ - 2)\tau n_+^{\gamma_+ + 1}}{n_+^4}. \tag{15}$$

Let us consider the Poisson equation (8). Using the rules of differentiation of the complex function

$$\frac{d^2\varphi}{d\xi^2} = \frac{d\varphi}{dn_+}\frac{d^2 n_+}{d\xi^2} + \frac{d^2\varphi}{dn_+^2}\Big(\frac{dn_+}{d\xi}\Big)^2,$$

and expressions (12), (13), and (15), we reduce the Poisson equation (8) to an autonomous second-order differential equation for $n_+(\xi)$:

$$-\frac{\mathrm{M}^2-\tau n_+^{\gamma_++1}}{n_+^3}\,\frac{d^2n_+}{d\xi^2}+\frac{3\mathrm{M}^2+(\gamma_+-2)\tau n_+^{\gamma_++1}}{n_+^4}\Big(\frac{dn_+}{d\xi}\Big)^2$$

$$=n_+-\Big\{1-(\gamma_--1)\Big[\frac{\tau}{\gamma_+-1}\,(n_+^{\gamma_+-1}-1)+\frac{\mathrm{M}^2}{2}\Big(\frac{1}{n_+^2}-1\Big)\Big]\Big\}^{1/(\gamma_--1)}. \tag{16}$$

The order of the equation can be lowered by performing the substitution $p(n_+)=dn_+/d\xi$. As a result, we obtain the differential Bernoulli equation [15, Sec. 1.1.5]

$$\frac{dp}{dn_+}=f_1(n_+)p+f_N(n_+)p^N \tag{17}$$

with the components

$$N=-1,\qquad f_1(n_+)=\frac{1}{n_+}\,\frac{3\mathrm{M}^2+(\gamma_+-2)\tau n_+^{\gamma_++1}}{\mathrm{M}^2-\tau n_+^{\gamma_++1}},$$
$$f_{-1}(n_+)=-\frac{n_+^4-n_+^3\{1-(\gamma_--1)[\tau(n_+^{\gamma_+-1}-1)/(\gamma_+-1)+\mathrm{M}^2(1/n_+^2-1)/2]\}^{1/(\gamma_--1)}}{\mathrm{M}^2-\tau n_+^{\gamma_++1}}. \tag{18}$$

Using the solution of the Bernoulli equation given in [15], we write the general solution with the integration constant C_1 for Eq. (17) with the components (18):

$$p^2=\frac{n_+^6}{(-\mathrm{M}^2+\tau n_+^{\gamma_++1})^2}\Bigg[C_1+2\int\frac{-\mathrm{M}^2+\tau n_+^{\gamma_++1}}{n_+^3}$$
$$\times\Bigg(n_+-\Big\{1-(\gamma_--1)\Big[\frac{\tau}{\gamma_+-1}\,(n_+^{\gamma_+-1}-1)+\frac{\mathrm{M}^2}{2}\Big(\frac{1}{n_+^2}-1\Big)\Big]\Big\}^{1/(\gamma_--1)}\Bigg)\,dn_+\Bigg]. \tag{19}$$

The integral of the Bernoulli equation (19) leads to an exact expression for $n_+(\xi)$ in implicit form with the integration constant C_2:

$$\xi+C_2=\int p(n_+)\,dn_+. \tag{20}$$

Together with (19), this expression is the exact solution of the problem of the nonlinear ion acoustic wave structure.

3. Numerical Example. The obtained exact solution of the problem of the nonlinear ion acoustic wave profile (19), (20) is very bulky. Therefore, it needs to be supplemented by a numerical example that allows one to study the features of the wave with the adiabatic temperature variation in the wave taken into account.

Figure 2 shows calculated profiles of physical parameters in an ion acoustic wave for the two particular cases of subsonic periodic and supersonic solitary waves. The profiles were determined as follows: the profile of the ion concentration $n_+(\xi)$ was first determined as the solution of the initial equation (16), and the profile of the electrostatic potential $\varphi(\xi)$ was then determined from formula (12), the profile of the electron concentration $n_-(\xi)$ from the nondimensionalized formula (5), the profile of the difference $n_+(\xi)-n_-(\xi)$, proportional to the value of the spatial charge in the wave, and, finally, the profiles of the temperatures $T_\pm(\xi)$ were determined from the dimensionless formulas $T_+=\tau(\gamma_-/\gamma_+)n_+^{\gamma_+-1}$ and $T_-=n_-^{\gamma_--1}$.

From Fig. 2a it is evident that, in the given example, the wave profile is not harmonic and the fluctuation amplitudes of the ion concentration and electron temperature are much larger than the fluctuation amplitudes of the electron concentration and ion temperature, respectively. The same ratio of the amplitudes for a solitary wave is observed in Fig. 2b.

4. Solitary Wave Mode. One of the basic issues of ion acoustic theory is the possibility of the existence of a solitary wave mode (an ion acoustic soliton). The form of Eq. (16), which reduces to the Bernoulli equation with the parameter $N=-1$ [see (18)], allows one to easily determine the boundary of propagation of the solitary wave mode on which there is breaking of the wave. Indeed, the solution of the Bernoulli equation (19) has the form of the energy conservation law for a certain pseudo-oscillator in an external field; in this case, the right side of (19)

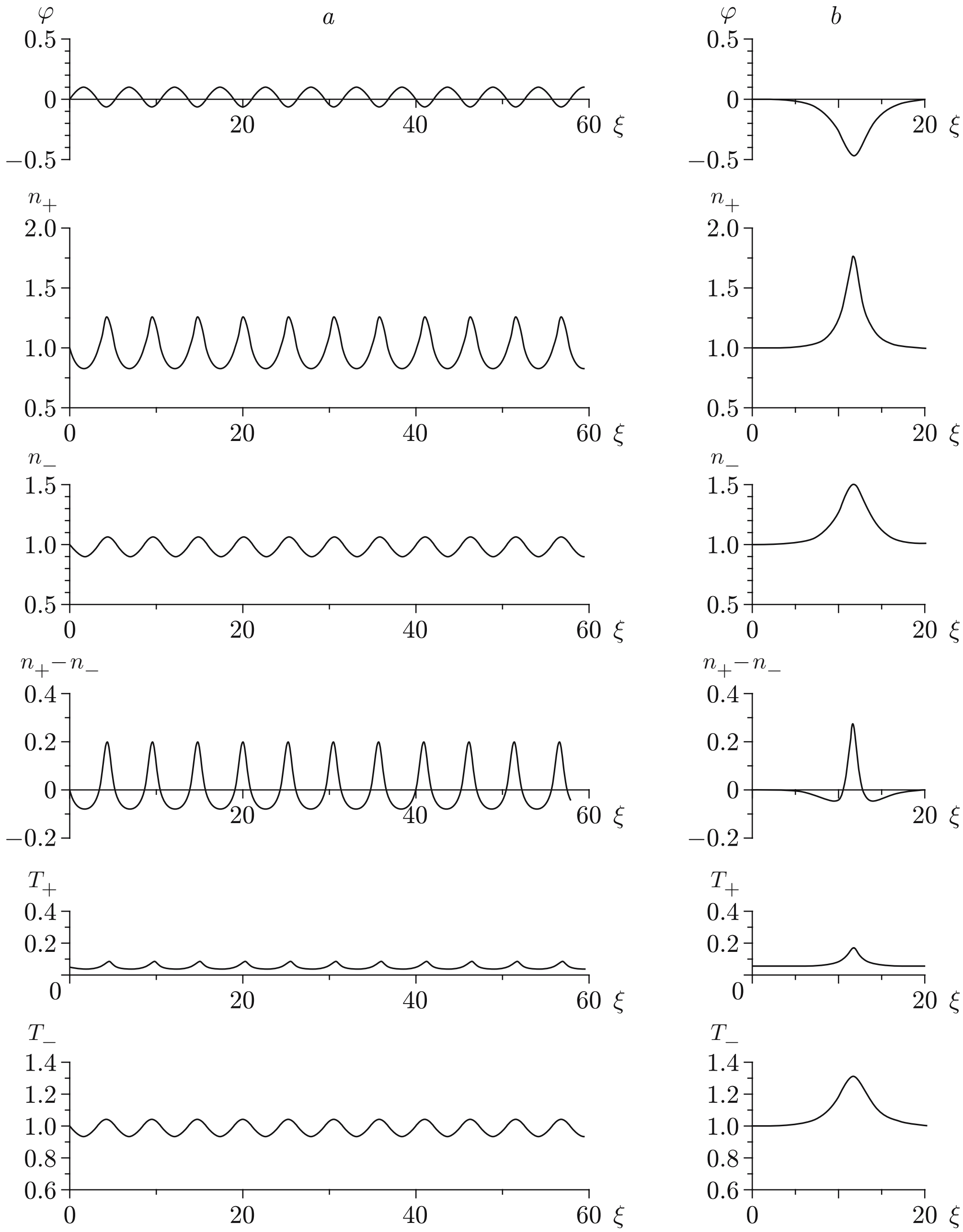

Fig. 2. Parameters in an ion acoustic wave ($\gamma_- = 3$, $\gamma_+ = 5/3$, and $\tau = 0.1$): (a) subsonic periodic wave ($\mathrm{M} = 0.7$); (b) supersonic periodic wave ($\mathrm{M} = 1.3$).

can play the role of a certain pseudo-potential $U(n_+)$ with a pseudo-coordinate n_+. The arbitrary constant C_1 can be chosen so that the condition $U(1) = 0$ is satisfied. As a result, we obtain

$$U(n_+) = \frac{2n_+^6}{(-\mathrm{M}^2 + \tau n_+^{\gamma_+ +1})^2} \int\limits_1^{n_+} \frac{-\mathrm{M}^2 + \gamma\tau n_+^{\gamma_+ +1}}{n_+^3}$$

$$\times \left(n_+ - \left\{ 1 - (\gamma_- - 1)\Big[\frac{\tau}{\gamma_+ - 1}\,(n_+^{\gamma_+ -1} - 1) + \frac{\mathrm{M}^2}{2}\Big(\frac{1}{n_+^2} - 1\Big)\Big]\right\}^{1/(\gamma_- - 1)} \right) dn_+.$$

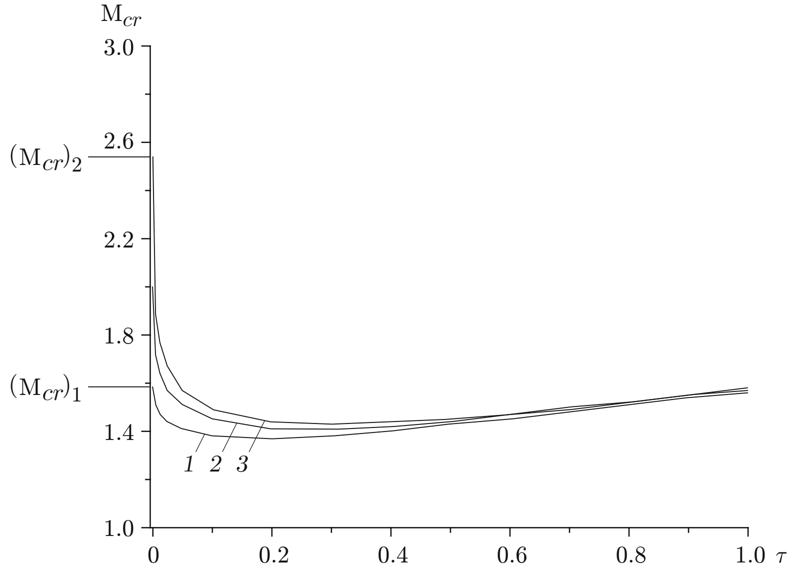

Fig. 3. Critical Mach number of solitary-wave breaking versus τ for various adiabatic exponents of the electron and ion gas components: $\gamma_\pm = 1$ (1), 2 (2), and 3 (3); $(M_{cr})_1 = 1.58$ [1–3] and $(M_{cr})_2 = 2.54$ [5].

Then, the equation of the boundary of the solitary-wave breaking $M = M_{cr}$ is written as

$$U(n_{+,\max}) > 0. \tag{21}$$

This implies that the extreme point of the curve of the pseudo-potential $U(n_+)$ is located not below the local maximum of $U(1)$. Finding M_{cr} from the condition (21) is possible only numerically.

Figure 3 shows curves of the critical Mach number of solitary-wave breaking versus τ for various adiabatic exponents of the electron and ion gas components $\gamma_\pm$ (the case $\gamma_+ = \gamma_-$ is considered). It turns out that these curves have a minimum at $\tau \approx 0.2$.

It should be noted that for large values of the exponents $\gamma_\pm$, the critical Mach numbers are large ($M_{cr} \geqslant 2$), whereas for $\gamma_\pm = 1$ and $\tau = 0$ (the electrons are isothermal and the ions are cold), $M_{cr} = 1.58$ [1–3]. For $\gamma_\pm = 3$ and $\tau \to 0$ (the electrons are adiabatic, and the ions cold), M_{cr} tends to the exact value $\sqrt{3(\sqrt{4/3}+1)} \approx 2.54$ found in [5].

In conclusion, we note that the present work complements the work of [5], in which a qualitative analysis of a similar system was performed without obtaining the exact solution. Thus, the construction of ion acoustic theory for a plasma using the gas-dynamic approach, which was began in [5], is completed.

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